Glauber critical dynamics: Exact solution of the kinetic Gaussian model

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In this paper, we have exactly solved Glauber's critical dynamics of the Gaussian model in three dimensions. Of course, it is much easier to apply in the low-dimensional case. The key steps are that we generalize the spin change mechanism from Glauber's single-spin flipping to single-spin transition and give a normalized version of the transition probability. We have also investigated the dynamical critical exponent and found surprisingly that the dynamical critical exponent is highly universal; that is, for one, two, and three dimensions they have the same value independent of spatial dimensionality in contrast to static (equilibrium) critical exponents. [S1063-651X(99)09902-X]

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I. INTRODUCTION

Irreversible dynamic systems exhibit complicated and interesting nonequilibrium phenomena near the critical point. The study of nonequilibrium statistical mechanics is much more difficult than equilibrium state due to the complexity. However, the interesting dynamic critical behaviors have been attracting a large number of researchers to work hard for many years.

Up to now, there is no general theory based on first principles to describe the dynamic problems. However, great progress has been made since the pioneering work completed by Glauber [1] and Kawasaki [2]. According to their theory, the time evolving of the order parameters is described by Markov processes with the Glauber single-spin flipping mechanism or Kawasaki exchange mechanism between two spins. Since then much attention has been paid to the study of critical dynamics. The research so far has been extended from the kinetic Ising model to the kinetic Potts model and from integer to fractional dimensions, in which many approximate methods such as Monte Carlo simulation, high-temperature series expansion, ϵ expansion, the bond-moving renormalization-group method, etc. have been applied [3–19].

Now let us turn to the master equation, a basic equation for treating critical dynamics. As we know, the key step for solving the master equation is the determination of the transition probability. Usually the transition probability between different states, i.e., different spin configurations of the system, is only determined in terms of the detailed balance condition. Since such a choice is not unique, some arbitrariness remains. For removing the arbitrariness, at least in part, we suggest a normalized transition probability; this means that in the unit time interval that the transition may occur or may not. We apply this point to the continuous spin (Gaussian) system and obtain exact solutions of one-, two-, and three-dimensional (1D, 2D, and 3D) kinetic Gaussian models.

This paper is organized as follows. In Sec. II, a single-spin transiting critical dynamics that suits arbitrary-spin systems is presented. As an application, the Gaussian model is treated in Sec. III. We not only obtain the exact solutions of 1D, 2D, and 3D kinetic Gaussian models, but also find that the dynamic critical exponent is highly universal. Finally, Sec. IV is devoted to the conclusion.

II. FORMALIZATION

For an irreversible dynamic system subjected to the time-dependent perturbation, once the perturbation is removed, the system will very slowly approach the equilibrium state because of the large-scale fluctuation near the critical point, which is what we call the critical slowing down phenomenon. We will attempt to explain using the critical dynamics, why the short-range interactions lead to long-time relaxation. Due to the complexity there have been, to our knowledge, no microscopic theories based on first principles so far; thus a suitable theoretical model will be quite important. As already mentioned, both the Glauber dynamics and Kawasaki dynamics have proven to be successful in many dynamic systems.

In this section, we give a brief review of Glauber's dynamics. Of course, we will give some improvement so that it can be applied to arbitrary-spin systems. For clarity, we start from the one-dimensional case, and then the formulation is easy to extend to two and three dimensions.

The 1D lattice-spin model we will discuss is a stochastic one. The spins of N fixed particles are represented as stochastic functions of time $\sigma_j(t)$, $(j=1,\ldots,N)$, which can be taken as discrete values (discrete-spin model) or continuous values (continuous-spin model), and made into transitions among these values. The transition, according to Glauber dynamics, can only change single-spin value each time, such as $\sigma_j(t)\to\hat{\sigma}_j(t)$ because of the interacting of the system with the heat reservoir. The transition probability $W_j(\sigma_j(t)\to\hat{\sigma}_j(t))$ from configuration $[\sigma_1(t),\sigma_2(t),\ldots,\sigma_j(t),\ldots,\sigma_N(t)]$ to configuration

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$[\sigma_1(t),\sigma_2(t),\ldots,\hat{\sigma}_j(t),\ldots,\sigma_N(t)]$, in general, depends on the momentary values of the neighboring spins as well as on the influence of the heat bath. For this reason statistical correlations exist between different spins. Therefore, it is necessary to deal with the entire N-spin system as a unit. The evolution of spin functions describing system form a Markov process of N discrete or continuous random variables with a continuous time variable as argument.

We introduce a probability distribution function $p(\sigma_1,\ldots,\sigma_N,t)$, which denotes the probability of spin system being in the state $(\sigma_1,\ldots,\sigma_N)$ at time t. Let $W_j(\sigma_j\to\hat{\sigma}_j)$ be the transition probability per unit time that the jth spin transits from one value σ_j to another possible value $\hat{\sigma}_j$, while the others remain fixed. Then, on the supposition of single-spin transition, we may write the time derivative of the function $P(\sigma_1,\ldots,\sigma_N,t)$ as

$$\frac{d}{dt}P(\{\sigma\},t)=\sum_j\sum_{\hat{\sigma}_j}[-W_j(\sigma_j\to\hat{\sigma}_j)p(\{\sigma\},t)+W_j(\hat{\sigma}_j\to\sigma_j)p(\{\sigma_{i\neq j}\},\hat{\sigma}_j,t)]. \tag{1}$$

This is a probability equation, in which the first term in the right-hand side of Eq. (1) denotes the decrease of the probability distribution function $P(\{\sigma\},t)$ per unit time due to the transition of the spin state from the initial value $\sigma_j,(j=1,2,\ldots,N)$ to various possible final values $\hat{\sigma}_j$, and the second term denotes the increase of the probability distribution function $P(\{\sigma\},t)$ per unit time due to the transition of the spin state from the various possible initial values $\hat{\sigma}_j,(j=1,2,\ldots,N)$ to final value σ_j. We shall refer to Eq. (1) as the master equation since its solution would contain the most complete description of the system available.

It is a most crucial step, obviously, that the transition probability must be determined before the master equation can be solved. Then, how can we determine the transition probability? For this problem, Glauber's theory leaves some leeway. However, inappropriate selection will probably make the problem difficult to solve. So, we hope to discover a more definite expression to apply the Glauber's theory to arbitrary-spin systems. Now we consider it in terms of both its mathematical and physical aspects. In mathematics, generally speaking, the probability must be ergodic and positive definite, and can be normalized; in physics, we often request that an equilibrium thermodynamic system satisfies the detailed balance condition. Based on these considerations, we can choose the spin transition probability $W_j(\sigma_j\to\hat{\sigma}_j)$ to satisfy the following conditions in order to ensure that the system is in a thermodynamic equilibrium state.

For the set $(S,\hat{S})$ composed of a subset S and its dual subset $\hat{S}$ in phase space, existing σ_j belonging to S and $\hat{\sigma}_j$ belonging to $\hat{S}$, we have the following:

(1) ergodicity,

$$\forall\sigma_j,\hat{\sigma}_j:\ W_j(\sigma_j\to\hat{\sigma}_j)\neq 0; \tag{2}$$

(2) positivity,

$$\forall\sigma_j,\hat{\sigma}_j:\ W_j(\sigma_j\to\hat{\sigma}_j)\geqslant 0; \tag{3}$$

(3) normalization,

$$\forall\sigma_j:\ \sum_{\hat{\sigma}_j}W_j(\sigma_j\to\hat{\sigma}_j)=1; \tag{4}$$

(4) detailed balance,

$$\forall\sigma_j,\hat{\sigma}_j:\ \frac{W_j(\sigma_j\to\hat{\sigma}_j)}{W_j(\hat{\sigma}_j\to\sigma_j)}=\frac{P_{eq}(\sigma_1,\ldots,\hat{\sigma}_j,\ldots,\sigma_N)}{P_{eq}(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_N)}, \tag{5}$$

in which

$$P_{eq}=\frac{1}{Z}\exp[-\beta\mathcal{H}(\{\sigma\})],\quad Z=\sum_{\{\sigma\}}\exp[-\beta\mathcal{H}(\{\sigma\})],$$

where P_{eq} is the equilibrium Boltzmann distribution function, Z is the partition function, and $\mathcal{H}(\{\sigma\})$ is the system Hamiltonian.

Although the spin transition probabilities are not determined uniquely by the above restriction conditions, there is less room within which to choose them. Furthermore, considering the fact that the transition of the individual spin depends merely on the momentary values of the neighboring spins as well as on the influence of the heat bath, we can assume that the transition probability from σ_j to $\hat{\sigma}_j$ depends only on the heat Boltzmann factor of the neighboring spins, i.e.,

$$W_i(\sigma_i\to\hat{\sigma}_i)\propto\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle ij\rangle}\sigma_j\right)\right],$$

or

$$W_i(\sigma_i\to\hat{\sigma}_i)=\frac{1}{Q_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle ij\rangle}\sigma_j\right)\right], \tag{6}$$

where $\Sigma_{<ij>}$ means that the summation for j is only related to the neighboring values of i. By means of the normalized condition (4), the normalized factor Q_i can be determined as

$$Q_i=\sum_{\hat{\sigma}_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle ij\rangle}\sigma_j\right)\right]. \tag{7}$$

Obviously, Q_i is independent of σ_i and is related to the temperature and neighboring spins.

Compared with Glauber's expression [1], Equation (6) is a normalized version of transition probability. As we know, the constant α in Glauber's expression is a free constant determined by the time scale. Actually, our expression is only a definite selection for constant α by extra restriction conditions and physical considerations.

Usually, we are interested in local magnetization and the spin-pair correlation, they are defined as follows [1]:

$$q_k(t)=\langle\sigma_k(t)\rangle=\sum_{\{\sigma\}}\sigma_kP(\{\sigma\},t), \tag{8}$$

$$\gamma_{kl}(t)=\langle\sigma_k(t)\sigma_l(t)\rangle=\sum_{\{\sigma\}}\sigma_k\sigma_lP(\{\sigma\},t). \tag{9}$$

According to definitions (8) and (9) and the master equation (1), and using the normalized condition (4), time-evolving equations of $q_k(t)$ and $\gamma_{kl}(t)$ can be derived (see Appendix A):

$$\frac{d}{dt}q_k(t)=-q_k(t)+\sum_{\{\sigma\}}\left(\sum_{\hat{\sigma}_k}\hat{\sigma}_k W_k(\sigma_k\to\hat{\sigma}_k)\right)P(\{\sigma\},t), \tag{10}$$

$$\frac{d}{dt}\gamma_{kl}(t)=-2\gamma_{kl}(t)+\sum_{\{\sigma\}}\left[\sigma_k\left(\sum_{\hat{\sigma}_l}\hat{\sigma}_l W_l(\sigma_l\to\hat{\sigma}_l)\right)+\sigma_l\left(\sum_{\hat{\sigma}_k}\hat{\sigma}_k W_k(\sigma_k\to\hat{\sigma}_k)\right)\right]P(\{\sigma\},t). \tag{11}$$

Similarly, time-evolving equation of equal-time multispin correlation functions can be further derived:

$$\begin{aligned}\frac{d}{dt}\langle\sigma_{i_1}(t)\sigma_{i2}(t)\cdots\sigma_{i_n}(t)\rangle &=-n\langle\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_n}\rangle\\&+\sum_{\{\sigma\}}\left\{\sum_{k=1}^{n}\left[\left(\prod_{j(\neq k)=1}^{n}\sigma_{i_j}\right)\right.\right.\\&\left.\left.\times\left(\sum_{\hat{\sigma}_{i_k}}\hat{\sigma}_{i_k}W_{i_k}(\sigma_{i_k}\to\hat{\sigma}_{i_k})\right)\right]\right\}P(\{\sigma\},t).\end{aligned} \tag{12}$$

Equations (1), (4)–(7), and (10)–(12) are the basic formulas of the single-spin transition-type critical dynamics, which suits one-dimensional arbitrary-spin systems.

All of these formulas can be readily extended to spin systems on square lattice and cubic lattice. The only corrections are changing subscript i into ij and ijk, respectively.

III. EXACT RESULTS

In this section, three examples of the application are given, including one-, two-, and three-dimensional kinetic Gaussian model.

Now we treat the kinetic Gaussian model. First of all we will introduce Gaussian model, then we will exactly solve the evolution of the local magnetization and equal-time spin-pair correlation function, and we will obtain the dynamical exponent z. Here, we will only give the solving process of the three-dimensional case in detail.

The Gaussian model, proposed by T. H. Berlin and M. Kac [20], at first in order to make an Ising model more tractable, is an continuous-spin model. Comparing it with the Ising model, besides having the same Hamiltonian form (three-dimensional case),

$$-\beta\mathcal{H}=k\sum_{i,j,k=1}^{N}\sum_{w}\sigma_{ijk}(\sigma_{i+w,j,k}+\sigma_{i,j+w,k}+\sigma_{i,j,k+w}), \tag{13}$$

where Σ_w means summation over near neighbors, there are two extensions: First, the spin σ_{ijk} can take any real value between $(-\infty,+\infty)$. Second, to prevent all spins from tending to infinity, a probability of finding a given spin between σ_{ijk} and $\sigma_{ijk}+d\sigma_{ijk}$ is assumed to be the Gaussian-type distribution

$$f(\sigma_{ijk})d\sigma_{ijk}=\sqrt{\frac{b}{2\pi}}\exp\left[-\frac{b}{2}\sigma_{ijk}^2\right]d\sigma_{ijk}, \tag{14}$$

where b is a distribution constant independent of temperature. Although it is an extension of the Ising model, the Gaussian model is quite different from the Ising model in terms of the properties of the phase transition. In the equilibrium case, on translational invariant lattices the Gaussian model was exactly solvable, and later as a starting point to study the unsolvable models it was also investigated with mean field theory and the momentum-space renormalization-group method [21,22]. Recently the Gaussian model on fractal lattices was studied by Li and Yang [23]. However, the critical dynamic problem of the continuous-spin model has never been investigated so far, to our knowledge.

We now proceed to treat the isotropic kinetic Gaussian model on the cube lattice. The system Hamiltonian and the spin distribution probability are Eqs. (13) and (14), respectively. In this case the spin transition probability can be expressed as

$$W_{ijk}(\sigma_{ijk}\to\hat{\sigma}_{ijk})=\frac{1}{Q_{ijk}}\exp\left[k\hat{\sigma}_{ijk}\sum_{w}(\sigma_{i+w,j,k}+\sigma_{i,j+w,k}+\sigma_{i,j,k+w})\right]. \tag{15}$$

Because the spin take continuous value, the summation for spin value turns into the integration

$$\sum_{\sigma}\to\int_{-\infty}^{\infty}f(\sigma)d\sigma; \tag{16}$$

then the normalized factor Q_{ijk} can be determined as

$$\begin{aligned}Q_{ijk}&=\int_{-\infty}^{\infty}d\hat{\sigma}_{ijk}f(\hat{\sigma}_{ijk})\times\exp\left[k\hat{\sigma}_{ijk}\sum_{w}(\sigma_{i+w,j,k}+\sigma_{i,j+w,k}+\sigma_{i,j,k+w})\right]\\&=\exp\left\{\frac{k^2}{2b}\left[\sum_{w}(\sigma_{i+w,j,k}+\sigma_{i,j+w,k}+\sigma_{i,j,k+w})\right]^2\right\},\end{aligned} \tag{17}$$

and the another useful combination formula can also be obtained:

$$\begin{aligned}\sum_{\hat{\sigma}_{ijk}}\hat{\sigma}_{ijk}W_{ijk}(\sigma_{ijk}\to\hat{\sigma}_{ijk})&=\int_{-\infty}^{\infty}\hat{\sigma}_{ijk}W_{ijk}(\sigma_{ijk}\to\hat{\sigma}_{ijk})f(\hat{\sigma}_{ijk})d\hat{\sigma}_{ijk}\\&=\frac{k}{b}\sum_{w}(\sigma_{i+w,j,k}+\sigma_{i,j+w,k}+\sigma_{i,j,k+w}).\end{aligned} \tag{18}$$

Substituting Eq. (18) into the following time-evolving equations of the local magnetization and the equal-time spin-pair correlation function,

$$\frac{d}{dt}q_{ijk}(t)=-q_{ijk}(t)+\sum_{\{\sigma\}}\left(\sum_{\hat{\sigma}_{ijk}}\hat{\sigma}_{ijk}w_{ijk}(\sigma_{ijk}\to\hat{\sigma}_{ijk})\right)P(\{\sigma\},t), \tag{19}$$

$$\frac{d}{dt}\gamma_{ijk;i'j'k'}(t)=-2\gamma_{ikj;i'j'k'}(t)+\sum_{\{\sigma\}}\left[\sigma_{ijk}\left(\sum_{\hat{\sigma}_{i'j'k'}}\hat{\sigma}_{i'j'k'}w_{i'j'k'}\times(\sigma_{i'j'k'}\to\hat{\sigma}_{i'j'k'})\right)+\sigma_{i'j'k'}\left(\sum_{\hat{\sigma}_{ijk}}\hat{\sigma}_{ijk}w_{ijk}(\sigma_{ijk}\to\hat{\sigma}_{ijk})\right)\right]P(\{\sigma\},t), \tag{20}$$

we get

$$\frac{d}{dt}q_{ijk}(t)=-q_{ijk}(t)+\frac{k}{b}\sum_{w}(q_{i+w,j,k}+q_{i,j+w,k}+q_{i,j,k+w}), \tag{21}$$

$$\frac{d}{dt}\gamma_{ijk;i'j'k'}(t)=-2\gamma_{ijk;i'j'k'}(t)+\frac{k}{b}\sum_{w}[\gamma_{ijk;i'+w,j',k'}(t)+\gamma_{ijk;i',j'+w,k'}(t)+\gamma_{ijk;i',j',k'+w}(t)+\gamma_{i+w,j,k;i'j'k'}(t)+\gamma_{i,j+w,k;i'j'k'}(t)+\gamma_{i,j,k+w;i'j'k'}(t)]. \tag{22}$$

In order to solve Eqs. (21) and (22) in the nearest-neighbor interaction case ($w=\pm 1$), we introduce two generating functions [1]:

$$F_1(\lambda_1,\lambda_2,\lambda_3,t)=\sum_{i,j,k=-\infty}^{\infty}\lambda_1^i\lambda_2^j\lambda_3^k q_{ijk}(t), \tag{23}$$

and

$$F_2(\lambda_1,\cdots,\lambda_6,t)=\sum_{i,j,k;i',j',k'=-\infty}^{\infty}\lambda_1^i\lambda_2^j\lambda_3^k\lambda_4^{i'}\lambda_5^{j'}\lambda_6^{k'}\gamma_{ijk;i'j'k'}(t); \tag{24}$$

then Eqs. (21) and (22) turn into the following equations with resect to F_1 and F_2, respectively:

$$\frac{d}{dt}F_1(\lambda_1,\lambda_2,\lambda_3,t)=\left[-1+\frac{k}{b}\sum_{i=1}^{3}(\lambda_i+\lambda_i^{-1})\right]F_1(\lambda_1,\lambda_2,\lambda_3,t), \tag{25}$$

$$\frac{d}{dt}F_2(\lambda_1,\cdots,\lambda_6,t)=\left[-2+\frac{k}{b}\sum_{i=1}^{6}(\lambda_i+\lambda_i^{-1})\right]F_2(\lambda_1,\ldots,\lambda_6,t). \tag{26}$$

Solving Eqs. (25) and (26), we get

$$F_1(\lambda_1,\lambda_2,\lambda_3,t)=F_1(\lambda_1,\lambda_2,\lambda_3,0)e^{-t}\exp\left[\frac{k}{b}\sum_{i=1}^{3}(\lambda_i+\lambda_i^{-1})t\right], \tag{27}$$

$$F_2(\lambda_1,\ldots,\lambda_6,t)=F_2(\lambda_1,\cdots,\lambda_6,0)e^{-2t}\exp\left[\frac{k}{b}\sum_{i=1}^{6}(\lambda_i+\lambda_i^{-1})t\right]. \tag{28}$$

In terms of a generating function of the Bessel functions of imaginary argument,

$$e^{x(\lambda+\lambda^{-1})/2}=\sum_{v=-\infty}^{\infty}\lambda^v I_v(x), \tag{29}$$

we obtain immediately the following exact solutions:

$$q_{ijk}(t)=e^{-t}\sum_{n,m,l=-\infty}^{\infty}q_{nml}(0)\times I_{i-n}\left(\frac{2k}{b}t\right)I_{j-m}\left(\frac{2k}{b}t\right)I_{k-l}\left(\frac{2k}{b}t\right), \tag{30}$$

$$\gamma_{ijk;i'j'k'}(t)=e^{-2t}\sum_{n,m,l;n',m',l'=-\infty}^{\infty}\gamma_{nml;n'm'l'}(0)\times I_{i-n}\left(\frac{2k}{b}t\right)I_{j-m}\left(\frac{2k}{b}t\right)I_{k-l}\left(\frac{2k}{b}t\right)\times I_{i'-n'}\left(\frac{2k}{b}t\right)I_{j'-m'}\left(\frac{2k}{b}t\right)I_{k'-l'}\left(\frac{2k}{b}t\right), \tag{31}$$

where $q_{nml}(0)$ and $\gamma_{nml;n'm'l'}(0)$, respectively, correspond to their initial values.

By using the asymptotic expansion expression of the first-kind imaginary argument Bessel function,

$$I_v(x) = \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{(-)^n (v,n)}{(2x)^n} + \frac{e^{-x+(v+1/2)\pi i}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{(v,n)}{(2x)^n},$$
$$(-\pi/2 < \arg x < 3\pi/2), \quad |x| \to \infty, \tag{32}$$

where

$$(v,n) = \frac{\Gamma(\frac{1}{2}+v+n)}{n!\Gamma(\frac{1}{2}+v-n)},$$

we can obtain the long-time asymptotic behavior of the local magnetization,

$$q_{ijk}(t) \sim \left(\frac{2k}{b}t\right)^{-3/2} e^{-(1-6k/b)t} \sum_{n,m,l=-\infty}^{\infty} q_{nml}(0) \sim \frac{1}{t^{3/2}} e^{-t/\tau}, \tag{33}$$

$$\tau = \frac{1}{1-6k/b}, \tag{34}$$

where τ is the relaxation time of the system. We know that the critical point of the Gaussian model is $k_c = J/k_\beta T_c = b/2d$, where d is the spatial dimension, and the correlation length critical exponent is $\nu = 1/2$ [22]. So, by means of the following dynamical scaling hypotheses

$$\xi \sim |T-T_c|^{-\nu}, \tag{35}$$

$$\tau \sim \xi^z, \tag{36}$$

the dynamic critical exponent z of the 3D kinetic Gaussian model can be obtained,

$$z = 2. \tag{37}$$

With the same treatment, we can easily solve the one- and two-dimensional kinetic Gaussian models. Ignoring the process of solution, we give only the following exact results:

(1) 1D case,

$$q_k(t) = e^{-t} \sum_{m=-\infty}^{\infty} q_m(0) I_{k-m}\left(\frac{2k}{b}t\right), \tag{38}$$

$$\gamma_{kl}(t) = e^{-2t} \sum_{n,m=-\infty}^{\infty} \gamma_{nm}(0) I_{k-n}\left(\frac{2k}{b}t\right) I_{l-m}\left(\frac{2k}{b}t\right), \tag{39}$$

$$\tau = \frac{1}{1-2k/b}, \tag{40}$$

$$z = 2. \tag{41}$$

(2) 2D case,

$$q_{nm}(t) = e^{-t} \sum_{k,l=-\infty}^{\infty} q_{k,l}(0) I_{n-k}\left(\frac{2k}{b}t\right) I_{m-l}\left(\frac{2k}{b}t\right), \tag{42}$$

$$\gamma_{mn;m'n'}(t) = e^{-2t} \sum_{i,j;i',j'=-\infty}^{\infty} \gamma_{ij;i'j'}(0) I_{m-i}\left(\frac{2k}{b}t\right) \times I_{n-j}\left(\frac{2k}{b}t\right) I_{m'-i'}\left(\frac{2k}{b}t\right) I_{n'-j'}\left(\frac{2k}{b}t\right) \tag{43}$$

$$\tau = \frac{1}{1-4k/b}, \tag{44}$$

$$z = 2. \tag{45}$$

IV. CONCLUSION

In this paper, we have suggested a normalized transition probability to solve the time evolution equations of the local magnetization and spin-pair correlation function. Our treatment can in part remove the arbitrariness in Glauber's dynamical theory, and makes it possible to exactly solve the time evolution equation.

Based on our treatment, we have exactly solved the kinetic Gaussian model, and given the details for solving the three-dimensional case. We have found, surprisingly, that the dynamical critical exponents have the same value independent of spatial dimension, which shows that the dynamical behavior has superuniversality, in contrast with static behavior. In fact, in the equilibrium phase transition the critical exponents are strongly dependent on dimensionality.

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APPENDIX: PROOFS OF EQS. (10)–(12)

According to definition (8) and using the master equation (1), we have

$$\begin{aligned}
\frac{d}{dt} q_k(t) &= \frac{d}{dt} \sum_{\{\sigma\}} \sigma_k P(\{\sigma\},t) \\
&= \sum_{\{\sigma\}} \sum_i \sum_{\hat{\sigma}_i} [-\sigma_k w_i(\sigma_i \to \hat{\sigma}_i) P(\{\sigma\},t) \\
&\quad + \sigma_k w_i(\hat{\sigma}_i \to \sigma_i) P(\{\sigma_{j\neq i}\},\hat{\sigma}_i,t)] \\
&= \sum_{\{\sigma\}} \sigma_k \sum_{i(i\neq k)} \sum_{\hat{\sigma}_i} [-w_i(\sigma_i \to \hat{\sigma}_i) P(\{\sigma\},t) \\
&\quad + w_i(\hat{\sigma}_i \to \sigma_i) P(\{\sigma_{j\neq i}\},\hat{\sigma}_i,t)] \\
&\quad + \sum_{\{\sigma\}} \sum_{\hat{\sigma}_k} [-\sigma_k w_k(\sigma_k \to \hat{\sigma}_k) P(\{\sigma\},t) \\
&\quad + \sigma_k w_k(\hat{\sigma}_k \to \sigma_k) P(\{\sigma_{j\neq k}\},\hat{\sigma}_k,t)].
\end{aligned} \tag{A1}$$

Looking at the first term $(i\neq k)$ following the last equality sign of Eq. (A1),

$$(i\neq k)\ \text{term} = \sum_{\{\sigma\}} \sigma_k \sum_{i(i\neq k)} \sum_{\hat{\sigma}_i} [-w_i(\sigma_i\to\hat{\sigma}_i) \times P(\{\sigma\},t) + w_i(\hat{\sigma}_i\to\sigma_i)P(\{\sigma_{j\neq i}\},\hat{\sigma}_i,t)]$$
$$= \sum_{\{\sigma_{j\neq i}\}} \sigma_k \sum_{i(i\neq k)} \left(-\sum_{\sigma_i,\hat{\sigma}_i} w_i(\sigma_i\to\hat{\sigma}_i)P(\{\sigma\},t) + \sum_{\sigma_i,\hat{\sigma}_i} w_i(\hat{\sigma}_i\to\sigma_i)P(\{\sigma_{j\neq i}\},\hat{\sigma}_i,t)\right), \quad \text{(A2)}$$

it is easy to see that this term equals to zero, as long as $\hat{\sigma}_i$ is exchanged with σ_i before summing for $\hat{\sigma}_i$ and σ_i. So the surplus term of Eq. (A1) is only the last term ($i=k$):

$$\frac{d}{dt}q_k(t) = \frac{d}{dt}\sum_{\{\sigma\}} \sigma_k P(\{\sigma\},t)$$
$$= \sum_{\{\sigma\}} \sum_{\hat{\sigma}_k} [-\sigma_k w_k(\sigma_k\to\hat{\sigma}_k)P(\{\sigma\},t) + \sigma_k w_k(\hat{\sigma}_k\to\sigma_k)P(\{\sigma_{j\neq k}\},\hat{\sigma}_k,t)]$$
$$= -\sum_{\{\sigma\}} \sigma_k \left(\sum_{\hat{\sigma}_k} w_k(\sigma_k\to\hat{\sigma}_k)\right) P(\{\sigma\},t) + \sum_{\sigma_1\cdots\sigma_k\hat{\sigma}_k\cdots\sigma_N} \sigma_k w_k(\hat{\sigma}_k\to\sigma_k)P(\{\sigma_{j\neq k}\},\hat{\sigma}_k,t)$$
$$= -\sum_{\{\sigma\}} \sigma_k P(\{\sigma\},t) + \sum_{\sigma_1\cdots\hat{\sigma}_k\sigma_k\cdots\sigma_N} \hat{\sigma}_k w_k(\sigma_k\to\hat{\sigma}_k)P(\{\sigma_{j\neq k}\},\sigma_k,t)$$
$$= -q_k(t) + \sum_{\{\sigma\}} \left(\sum_{\hat{\sigma}_k} \hat{\sigma}_k w_k(\sigma_k\to\hat{\sigma}_k)\right) P(\{\sigma\},t),$$

in which the normalized condition and the method of exchange of $\hat{\sigma}_i$ for σ_i were used. Hitherto, Eq. (10) has been proven exactly. As for the proof of the Eqs. (11) and (12), it is easily accessible via the same method and thus does not require further proof.

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